Part 1. Version A

In part 1, only the final answer is required. You do not need to show your calculations.

Exercise 1: *[3 Points], 1 for each answer*

Let X and Y be two independent normal random variables. We have $E[X] = 0$, $E[Y] = 2$, $Var(X) = 1$ and $Var(Y) = 15$. Round the answer from the table to the third decimal place where necessary.

1. $Var(X + Y) =$ $\boxed{16}$
2. $P(X \leqslant 1) =$ $\boxed{0.841}$
3. $P(\frac{X-Y-2}{2} \geqslant 2)$ $\boxed{0.159}$

Exercise 2 *[2 Points] All 4 correct: 2pt ; 3 correct: 1pt; 2 correct: 0.5pt; otherwise 0 pt.*

Let A and B be two events. $\bar{A}$ and $\bar{B}$ represent the complements of A and B respectively. We know that $P(A) = 0.7$, $P(B) = 0.4$ and $P(A \mid B) = 0.3$.

1. A and B are independent $\boxed{\text{False}}$
2. $P(A \cup B) = P(A) + P(B)$ $\boxed{\text{False}}$
3. $P(\overline{A \cup B}) = 0.15$ $\boxed{\text{False}}$
4. $P(\bar{A} \mid B) = 0.7$ $\boxed{\text{True}}$

Exercise 3: *[3 Points] 1 for each answer*

Let X and Y be two independent uniform continuous random variables. We have $S_X = (-2, 1)$ and $S_Y = (1, 3)$. Answer the following, **round your answers to 3 decimal places, where appropriate**:

1. $P(X \leqslant 0) =$ $\boxed{0.667}$
2. $E[X + Y] =$ $\boxed{1.5}$
3. What is the support of the product between X and Y (S_{XY})= $\boxed{(-6, 3)}$

Exercise 4: *[2 Points] 1 for each answer*

We throw an unbiased coin in the air 100 times. For each k, $X_k = 1$ if the result of the k^{th} throw was tails. We define $\bar{X}_n = \sum_{k=1}^{n} X_k$. Answer each question by clearly writing either **True** or **False inside the boxes**

1. $\bar{X}_{100}$ follows a Binomial distribution $\boxed{\text{True}}$

2. $\dfrac{\bar{X}_{100} - 50}{10}$ can be approximated by a standard normal random variable $N(0,1)$ $\boxed{\text{False}}$

Part 1. Version B

Exercise 1: *[2 Points] 1 for each answer*

We throw an unbiased dice 100 times. For each k, $X_k = 1$ if the result of the k^{th} throw was a 1. We define $\bar{X}_n = \sum_{k=1}^{n} X_k$. Answer each question by clearly writing either **True** or **False inside the boxes**

1. $Var(\bar{X}_{100}) \approx 13.889$ $\boxed{\text{True}}$

2. $36\dfrac{\bar{X}_{100} - \frac{500}{6}}{50}$ can be approximated by a standard normal random variable $N(0,1)$ $\boxed{\text{False}}$

Exercise 2 *[2 Points] All 4 correct: 2pt ; 3 correct: 1pt; 2 correct: 0.5pt; otherwise 0 pt.*

For each of the following statements, write clearly either **True** or **False** inside the boxes.

Let A and B be two events. $\bar{A}$ and $\bar{B}$ represent the complements of A and B respectively. We know that $P(A) = 0.7$, $P(B) = 0.4$ and $P(A \mid B) = 0.3$.

1. A and B are independent $\boxed{\text{False}}$

2. $P(A \cup B) = P(A) + P(B)$ $\boxed{\text{False}}$

3. $P(\overline{A \cup B}) = 0.15$ $\boxed{\text{False}}$

4. $P(\bar{A} \mid B) = 0.7$ $\boxed{\text{False}}$

Exercise 3: *[3 Points] 1 for each answer*

Let X and Y be two independent uniform discrete random variables. We have $S_X = \{-3, -2, -1, 0, 1, 2\}$ and $S_Y = \{-1, 0, 1, 2, 3, 4, 5\}$. Answer the following, **round your answers to 3 decimal places, where appropriate**:

1. $P(X \leqslant 0) =$ $\boxed{\frac{4}{6} \approx 0.667}$

2. $E[X+Y] =$ $\boxed{1.5}$

3. What is the support of the product between X and Y
$(S_{XY}) = \{-15, -12, -9, -8, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 8, 10\}$

Exercise 4: *[3 Points], 1 for each answer*

Let X and Y be two independent normal random variables. We have $E[X] = 1$, $E[Y] = 2$, $Var(X) = 16$, $Var(Y) = 4$. Round the answer from the table to the third decimal place where necessary.

1. $P(\frac{Y-2}{2} \geqslant 0.5) =$ both 0.308 and 0.309 are accepted
2. $P(X \leqslant 3) =$ both 0.691 and 0.692 are accepted

3. $Var(3(X - Y)) = \boxed{180}$

Part 2.

Exercise 5:[4 Points]

We throw two fair six sided die at the same time. Let X_1 be the result of the first dice and X_2 be the result of the second dice. Let $X = \frac{7X_1}{2} - X_1 X_2$.

1. Calculate $E[X]$. *Round your answer to the third decimal place* ***[2 Points]***

   First compute $E[X_1]$:

$$\begin{aligned} E[X_1] &= \sum_k k P(X_1 = k) \\ &= \sum_{k=1}^{6} \frac{k}{6} \\ &= \frac{21}{6} = 3.5 \end{aligned}$$

   Then use linearity of expectation

$$\begin{aligned} E[X] &= \frac{7}{2} E[X_1] - E[X_1 X_2] \\ &= \frac{7}{2} E[X_1] - E[X_1]E[X_2] \textbf{ by independence} \\ &= \frac{7^2}{4} - \frac{7^2}{2^2} = 0 \end{aligned}$$

2. Calculate $Cov(X, X_1)$. Are X_1 and X independent? *Round your answer to the third decimal place* ***[2 Points]***

$$Cov(X, X_1) = E[XX_1] - E[X]E[X_1].$$

Because $E[X] = 0$,

$$\begin{aligned} Cov(X, X_1) = E[XX_1] &= \frac{7}{2}E[X_1^2] - E[X_1^2]E[X_2] \text{ \textbf{by independence}} \\ &= \frac{7}{2}E[X_1^2] - \frac{7}{2}E[X_1^2] = 0 \end{aligned}$$

If you had a different value for $E[X_1]$, need to compute $E[X_1^2]$:

$$\begin{aligned} E[X_1^2] &= \sum_{k=1}^{6} \frac{k^2}{6} \\ &= \frac{91}{6} \end{aligned}$$

Then

$$Cov(X, X_1) = \frac{7 \times 91}{2} - \frac{91E[X_1]}{6} - E[X]E[X_1].$$

X and X_1 are not independent: X can take the value -15 if $X_1 = X_2 = 6$. Given that $X_1 = 1$, X can never take the value -15.

In other words, $P(X = -15) > 0$ but $P(X = -15 \mid X_1 = 1) = 0$.

Note: Two variables can be independent even if one is used in the definition of the other. If $X \sim \mathcal{N}(0, 1)$ and $P(Y = 1) = P(Y = -1) = 0.5$. Then Y and $X \times Y$ are independent. In this case, the value of Y does not tell you anything about the value of $X \times Y$. On the other hand, X and $Y \times X$ have zero covariance but they are dependent: if $X = 2$, then $Y \times X$ is 2 or -2 with probability 1.

Exercise 6: *[8 Points]*

The continuous random variable X has a probability density function (pdf) f_X given by

$$f_X(x) = \frac{e^{-x}}{1 - e^{-2}} \text{ for } 0 \leqslant x \leqslant 2$$

.

1. Find the numerical value of $P(X < 1)$. ***[2 Points]***

$$\begin{aligned} P(X < 1) &= \int_0^1 \frac{e^{-x}}{1 - e^{-2}} dx \\ &= \frac{1}{1 - e^{-2}} [-e^{-x}]_0^1 \\ &= \frac{1 - e^{-1}}{1 - e^{-2}} \\ &\approx 0.731 \end{aligned}$$

2. Calculate $E[X]$ and $E[e^X]$ ***[3 Points]***

$$\begin{aligned}
E[X] &= \int_0^2 \frac{xe^{-x}}{1-e^{-2}}dx \\
&= \frac{1}{1-e^{-2}}([-xe^{-x}]_0^2 - \int_0^2 e^{-x}dx) \\
&= \frac{1}{1-e^{-2}}(-2e^{-2} - [-e^{-x}]_0^2 \\
&= \frac{1-3e^{-2}}{1-e^{-2}} \\
&\approx 0.69
\end{aligned}$$

$$\begin{aligned}
E[e^X] & \int_0^2 \frac{e^x e^{-x}}{1-e^{-2}}dx \\
&= \int_0^2 \frac{1}{1-e^{-2}}dx \\
&= \frac{2}{1-e^{-2}} \\
&\approx 0.135
\end{aligned}$$

3. Let $Y =\mid X-1 \mid$, where $\mid \cdot \mid$ is the absolute value. Determine the range S_Y and the probability density function f_Y.***[3 Points]***

   Absolute value is non-negative and the distance between X and 1 is bounded by 1. This means $S_Y = (0,1)$ (or $[0,1]$).

   In order to get the PDF we start with the CDF:

$$\begin{aligned}
F_y(y) = P(Y \leqslant y) &= P(\mid X-1 \mid \leqslant y) \\
&= P(1-y \leqslant X \leqslant 1+y) \\
&= \int_{1-y}^{1+y} \frac{e^{-x}}{1-e^{-2}}dx \\
&= \frac{1}{1-e^{-2}}[-e^{-x}]_{1-y}^{1+y} \\
&= \frac{e^{-1+y} - e^{-1-y}}{1-e^{-2}}
\end{aligned}$$

   Then we take the derivative of F_y :

$$f_y(y) = F_y'(y) = \frac{1}{1-e^{-2}}(e^{-1+y} + e^{-1-y})$$

Exercise 7: *[8 Points]*

A deck is composed of 40 cards numbered from 1 to 10. Each rank from 1 to 10 has 4 indistinguishable cards (there are four 1, four 2, etc...). We pick five cards uniformly at random.

1. What is the probability that exactly three cards among the five have the same rank? ***[2 Points]***

   We need to compute the number of favorable outcomes over the number of total possible outcomes

   In total, there are $\binom{40}{5}$ possible set of 5 cards.

   There are 10 posibilities for the rank of the first card. Once the rank has been chosen, there are $\binom{4}{3}$ possibilities to choose from the cards of the given rank.

   Then there are $\binom{36}{2}$ possibilities for the remaining cards (they can't be of the rank of the first three).

   Let's call S the event "exactly three cards among the five have the same rank".

$$P(S) = \frac{10\binom{4}{3}\binom{36}{2}}{\binom{40}{5}}.$$

   Alternatively:

$$P(S) = \sum_{k=1}^{10} P(S \text{ and the three cards have rank } k\ ) = 10P(S \text{ and the three cards have rank } 1\ ).$$

   Here we can use hypergeometric for $P(S$ and the three cards have rank 1 $)$ to get the same result. $N = 40$, $n = 5$, $R = 4$, $k = 3$.

2. Given that three of the cards are 1, 2 and 3, What is the probability that all 5 cards are consecutive numbers? ***[3 Points]***

   Let's call events A: "three of the cards are 1, 2 and 3"

   and B: "all 5 cards are consecutive numbers".

   We get

$$P(B \mid A) = \frac{P(B \cap A)}{P(A)}.$$

   The event $B \cap A$ is the event C: "the cards drawn are 1, 2, 3, 4 and 5". It is the only possibility for 5 cards to all be in succession if 1, 2 and 3 are fixed.

$$P(B \cap A) = \frac{4^5}{\binom{40}{5}} :$$

   for eack rank from 1 to 5 there are 4 possibilities for which card of that rank we draw, again divided by the total number of possible sets of 5 cards. And

$$P(A) = \frac{4^3\binom{37}{2}}{\binom{40}{5}}$$

   There are four possibilities when drawing a card of a given rank. Then the remaining 2 cards can be freely chosen among the 37 remaining cards, giving $\binom{37}{2}$ possibilities. Then,

$$P(B \mid A) = \frac{4^5\binom{40}{5}}{4^3\binom{37}{2}\binom{40}{5}} = \frac{4^2}{\binom{37}{2}} (= \frac{32}{37 \times 36})$$

3. Compute the expectation of the sum of the ranks of all five cards. ***[3 Points]***

   Let's X_k the rank of card k.

   For all k and j, $P(X_k = j) = 0.1$.

   Which means $E[X_1] = 0.1 \sum_{i=1}^{10} k = 5.5$.

   Then

$$
\begin{aligned}
E[\sum_{k=1}^{5} X_k] &= \sum_{k=1}^{5} E[X_k] \\
&= 5E[X_1] \\
&= 27.5
\end{aligned}
$$